An approximation procedure for the Zakharov - Shabat scattering problem for real single-humped potentials

1996 J. Phys. A: Math. Gen. 29 2493

(http://iopscience.iop.org/0305-4470/29/10/027)

J. Phys. A: Math. Gen. **29** (1996) 2493–2498. Printed in the UK

An approximation procedure for the Zakharov–Shabat scattering problem for real single-humped potentials

M Desaix†‡, D Anderson†, M Lisak† and M L Quiroga-Teixeiro†

† Institute for Electromagnetic Field Theory, Chalmers University of Technology, S-412 96 Göteborg, Sweden

‡ The University of Borås, Department of Engineering Sciences, Allégatan 1, Box 874, S-501 15 Borås, Sweden

Received 19 June 1995, in final form 30 November 1995

Abstract. A procedure is presented for finding simple approximations of the discrete eigenvalues of the Zakharov–Shabat scattering problem corresponding to the nonlinear Schrödinger equation. The approximation is in the form of an interpolation formula which combines results for small eigenvalues, obtained by a direct variational approach, and for large eigenvalues, obtained by the Bohr–Sommerfeld quantization rule.

The Zakharov–Shabat scattering problem plays a central role in the solution procedure of several important evolution equations describing the nonlinear propagation of wave pulses, e.g. the nonlinear Schrödinger equation, the sine–Gordon equation and the modified Korteweg–de Vries equation [1]. These equations are universal in the sense that they describe basic physical problems which appear in many different areas of physics. The analytical tool used to solve these nonlinear evolution equations is the inverse scattering transform method; the solution of the concomitant scattering problem [2] provides the discrete eigenvalues which yield information about the soliton properties of the solution. Although the problem of finding the discrete eigenvalues is linear, the solution is not available in closed form except for special cases. Consequently, there is great interest for both numerical [3, 4] and approximate analytical schemes for determining the eigenvalues of the Zakharov–Shabat scattering problem. Several approximate studies of the Zakharov–Shabat scattering problem have used the WKB method (cf [5, 6]). However, recently a new approach based on direct variational schemes has been suggested independently by two groups [7–9]. In this paper, we present a procedure for finding simple approximations of discrete eigenvalues of the Zakharov–Shabat scattering problem for the nonlinear Schrödinger equation. The procedure is based on a direct variational approach and the Bohr–Sommerfeld quantization rule.

The Zakharov–Shabat scattering problem reads

$$\frac{\mathrm{d}v_1}{\mathrm{d}x} = -\mathrm{i}\zeta v_1 + q(x)v_2 \qquad \text{and} \qquad \frac{\mathrm{d}v_2}{\mathrm{d}x} = -q^*(x)v_1 + \mathrm{i}\zeta v_2$$
$$v_1 \to \exp(-\mathrm{i}\zeta x) \qquad v_2 \to 0 \qquad x \to -\infty \tag{1}$$

0305-4470/96/102493+06$19.50

where v_1 and v_2 are the Jost functions, ζ is the eigenvalue and $q(x)$ is the potential corresponding to the initial pulse form. The Zakharov–Shabat equations can be reduced to a single second-order equation for v_1, by eliminating v_2. This yields

$$v_{1xx} - \frac{q_x}{q} v_{1x} + \left(\zeta^2 + |q|^2 - \mathrm{i}\zeta \frac{q_x}{q}\right) v_1 = 0 \qquad (2)$$

where the subscript x denotes differentiation with respect to x. In the subsequent analysis we will assume the potential $q(x)$ to be a real-valued single-humped function with a maximum A for $x = 0$, i.e. $q(0) \equiv A$. For smooth potentials and large values of A, the terms proportional to q_x/q in equation (2) can be neglected, as shown in [10]. This yields an ordinary Schrödinger equation for which we can use the Bohr–Sommerfeld quantization rule to determine the eigenvalues $\zeta = \mathrm{i}\eta$:

$$\int_{x_1}^{x_2} \sqrt{q^2(x) - \eta^2}\, \mathrm{d}x = \pi(n - \tfrac{1}{2}) \qquad n = 1, 2, \ldots. \qquad (3)$$

The limits of integration in equation (3), i.e. the turning points, are given by the two roots of the equation $q(x) = \eta$. The case $n = 1$ corresponds to the largest eigenvalue whereas larger integers correspond to higher-order modes. This so-called quasiclassical approach was suggested already in the original classical paper by Zakharov and Shabat [11]. Although (3) is primarily an approximation valid for large A, it is interesting to note that it is also exact for $\eta = 0$, since in this limit it reduces to the well known threshold condition for soliton generation [2, 12], namely

$$\int_{-\infty}^{\infty} |q(x)|\, \mathrm{d}x = \pi\left(n - \tfrac{1}{2}\right). \qquad (4)$$

This implies that (3) can be expected to give a good approximation for all eigenvalues as long as the potential is smooth enough, a fact which does not seem to have been emphasized before. We illustrate this property of the Bohr–Sommerfeld quantization condition by considering the case of a Gaussian potential $q = A\exp(-x^2)$ which has been extensively studied in connection with pulse propagation in optical fibres. The result for the eigenvalue η as a function of A, obtained by a numerical integration of the integral in (3), is shown in figure 1 and compared with a result of a numerical solution of the original Zakharov–Shabat scattering problem (equation (1)). The agreement is seen to be very good.

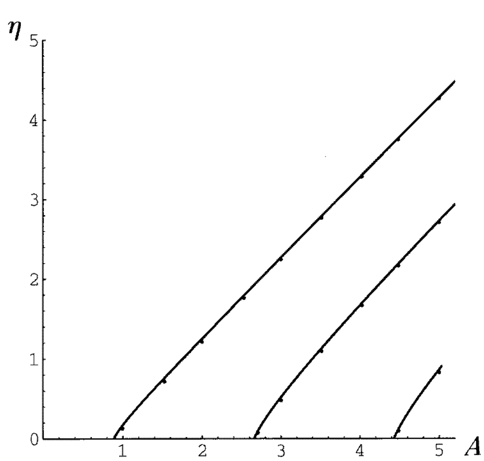

Figure 1. The imaginary part, η, of the discrete eigenvalue as a function of the amplitude A for the potential $q = A\exp(-x^2)$. The full curve represents a numerical solution of the Bohr–Sommerfeld quantization rule, equation (3), whereas the circles represent a full numerical solution of the Zakharov–Shabat eigenvalue problem, equation (1).

However, for less smooth potentials, predictions based on equation (3) can be expected to be less accurate. This is clearly seen in the extensive investigation by Kaup [6], where equation (3) has been used to obtain approximations for the eigenvalues in the case of a rectangular potential where the quantization integral in (3) can be performed analytically to give explicit expressions for the eigenvalue as a function of the height of the potential. The qualitative agreement between the predictions of (3) and a full numerical solution of the Zakharov–Shabat problem is found to be good, but the quantitative agreement is less so, in particular for small eigenvalues. Even for large eigenvalues there is a systematic shift between the analytical and numerical curves for $\eta = \eta(A)$, which is due to the fact that the terms containing q_x/q cannot be neglected in (2) for the rectangular box potential since this particular potential has an infinite slope at the edge of the pulse.

The problem of the rectangular potential was reconsidered in [9] using a new approach based on a direct variational method involving trial functions and a subsequent Ritz optimization. This variational analysis resulted in a simple analytical expression for the eigenvalue, in excellent agreement with the numerical solution.

A disadvantage of the Bohr–Sommerfeld quantization formula, equation (3), is the fact that the corresponding solution for the eigenvalue, in general, is complicated and has to be obtained by numerical integration. To remedy this situation we suggest a new approach which combines the merits of the Bohr–Sommerfeld formula and the variational approach.

Let us consider a single-humped and, for simplicity, symmetric potential involving a single scale parameter A, i.e. $q(x) = AQ(x)$. The Bohr–Sommerfeld formula, equation (3), then yields

$$2A \int_0^{x_2} \sqrt{Q^2(x) - Q^2(x_2)}\, \mathrm{d}x = \pi\left(n - \tfrac{1}{2}\right) \tag{5}$$

where we have used $\eta = q(x_2) = AQ(x_2)$. The right-hand side of (5) is finite; we must therefore require that $x_2 \to 0$ as $A \to \infty$. The limits of integration are hence small for large amplitudes and it is legitimate to use an expansion for $Q(x)$ valid for small x. In this paper we will focus on potentials with a parabolic shape in the central parts. Many interesting pulse shapes are of this type. However, it should be emphasized that the analysis can be carried out for other pulses with small modifications. Inserting the expansion $Q(x) \approx (1 - bx^2)$ in equation (5) yields

$$4\frac{Abx_2^2}{\sqrt{2b}} \int_0^1 \sqrt{(1-t^2)\left(1 - \frac{bx_2^2}{2}(1+t^2)\right)}\, \mathrm{d}t = \pi\left(n - \tfrac{1}{2}\right). \tag{6}$$

The integral approaches $\pi/4$ as $x_2 \to 0$, and the eigenvalue is given by $\eta = A(1 - bx_2^2)$. We therefore asymptotically obtain a straight-line solution, $\eta = A - \Delta$ where $\Delta = (n - \frac{1}{2})\sqrt{2b}$. This straight line can be expected to give a good approximation of the asymptotic behaviour of $\eta = \eta(A)$ for large eigenvalues. In the complementary limit of small eigenvalues an accurate straight-line approximation of $\eta = \eta(A)$ was found in [9] using a direct variational approach. The result is

$$\eta = k(A - A_{\mathrm{c}}) \tag{7}$$

where the critical amplitude, A_{c}, corresponding to $\eta = 0$, and the initial slope, k, of the curve $\eta = \eta(A)$ are given, respectively, by

$$A_{\mathrm{c}} = \frac{\frac{1}{2}\pi(2n-1)}{\langle Q \rangle} \tag{8}$$

Table 1. Critical amplitude A_c, initial slope k, and asymptotic offset Δ, according to equations (8), (9) and (6), respectively, for the potentials $q = A\exp(-x^2)$ and $q = A\,\mathrm{sech}^3(x)$.

	$q(x) = A\exp(-x^2)$			$q(x) = A\,\mathrm{sech}^3(x)$		
Eigenmode n	Critical amplitude A_c	Initial slope k	Asymptotic offset Δ	Critical amplitude A_c	Initial slope k	Asymptotic offset Δ
1	$\sqrt{\pi}/2$	1.190	$1/\sqrt{2}$	1	1.114	$\sqrt{3}/2$
2	$3\sqrt{\pi}/2$	1.563	$3/\sqrt{2}$	3	1.294	$3\sqrt{3}/2$
3	$5\sqrt{\pi}/2$	1.728	$5/\sqrt{2}$	5	1.352	$5\sqrt{3}/2$

and

$$k = \frac{\langle Q\rangle}{\left\langle \sin\left[2A_c\int_{-\infty}^{x} Q(x')\,\mathrm{d}x'\right]\right\rangle} \tag{9}$$

where Q is the normalized amplitude, i.e. $q(x) = AQ(x)$, and $\langle\cdots\rangle$ denotes integration with respect to x from minus infinity to plus infinity.

For the case of the classical scattering potential $q = A\,\mathrm{sech}(x)$, the straight-line approximations given by (6) and (7) coincide and in fact are equal to the exact result, i.e. $\eta = A - (n - \frac{1}{2})$ (cf the appendix). However, this coincidence of the asymptotic lines for small and large amplitudes, respectively, is peculiar to the sech-shaped potential. For other potentials, $q(x)$, equations (6) and (7) in general yield two different straight lines. It is then natural to use an interpolation formula which smoothly connects the two lines. One possible choice is to take $\eta = \eta(A)$ as the ratio of a second-order and a first-order polynomial in A and to determine the four independent parameters of the polynomials so as to obtain the correct asymptotic behaviour as $A \to A_c$ and $A \to \infty$, respectively. This yields

$$\eta = \frac{(A - A_c)[A - A_c + k(\Delta - A)]}{(A - A_c)(1 - k) + \Delta - A_c} \tag{10}$$

which represents a smooth transition between the two straight lines provided $(\Delta - A_c)/(1 - k) > 0$.

In order to illustrate the usefulness of equation (10) we consider two further examples: (i) the previously mentioned Gaussian pulse $q = A\exp(-x^2)$, and (ii) $q = A\,\mathrm{sech}^3(x)$, which has attracted attention lately in connection with nonlinear optical fibre loop mirrors [13]. Table 1 summarizes the results for the critical amplitude A_c, the initial slope k, and the asymptotic offset Δ for these pulses for the first three eigenmodes.

Figures 2 and 3 compare the predictions of the interpolation formula for the eigenvalues with numerical solutions of the Zakharov–Shabat scattering equations, for the potentials $q = A\exp(-x^2)$ and $q = A\,\mathrm{sech}^3(x)$, respectively. The agreement is seen to be very good.

In conclusion, we have derived a simple and useful analytical expression for the discrete eigenvalues of the Zakharov–Shabat scattering problem for real single-humped potentials, $q(x)$, which have a parabolic shape for $x \ll 1$. Application to the cases $q(x) = A\,\mathrm{sech}(x)$, $q(x) = A\,\mathrm{sech}^3(x)$ and $q(x) = A\exp(-x^2)$ has been shown to yield results which are in excellent agreement with numerical solutions of the Zakharov–Shabat scattering equations. We think that the variational approach should also be a useful tool for finding the discrete eigenvalues of the Zakharov–Shabat problem for other types of potentials. Furthermore, it should be possible to extend the approach to the corresponding scattering problems for other nonlinear evolution equations solvable by the inverse scattering technique.

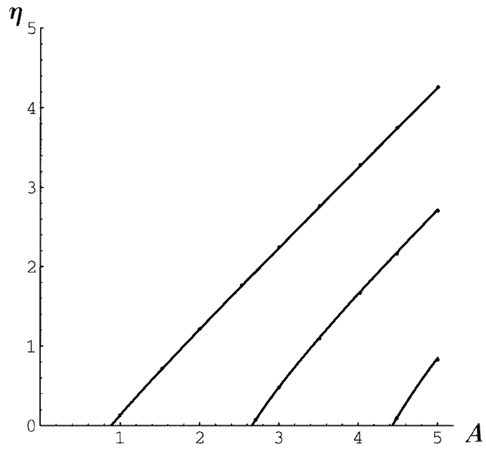

Figure 2. The imaginary part, η, of the discrete eigenvalue as a function of the amplitude A for the potential $q = A\exp(-x^2)$. The full curve is the approximation according to equation (9) whereas the circles represent a numerical solution of the Zakharov–Shabat eigenvalue problem (equation (1)).

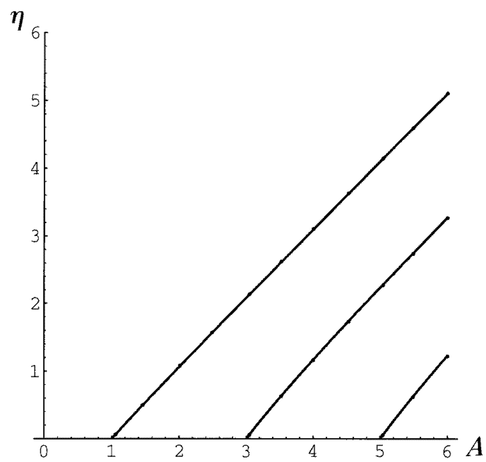

Figure 3. Same as figure 2 but for the potential $q = A\,\mathrm{sech}^3(x)$.

Acknowledgments

Helpful discussions with Dr M Lundin on the train between Borås and Gothenburg are gratefully acknowledged.

Appendix. The potential $q = A\,\mathrm{sech}(x)$

The classical scattering potential, $q = A\,\mathrm{sech}(x)$ has the remarkable feature that the solution to the corresponding Zakharov–Shabat scattering problem is very simple, i.e. the real part of the discrete eigenvalue is zero and the imaginary part, η, varies linearly with respect to the amplitude A.

In [9], we use a variational approach for obtaining the discrete eigenvalues of the Zakharov–Shabat scattering problem. For a particularly simple choice of test functions (in fact the eigenfunctions corresponding to the eigenvalue zero) we obtain a straight-line dependence between the eigenvalue and the amplitude, $\eta = k(A - A_c)$, where the critical amplitude A_c and the slope k are given by equations (8) and (9), respectively. For the potential $Q(x) = \mathrm{sech}(x)$ we readily obtain

$$A_c = \frac{\frac{1}{2}\pi(2n-1)}{\int_{-\infty}^{\infty}\mathrm{sech}(x)\,\mathrm{d}x} = \frac{\frac{1}{2}\pi(2n-1)}{\pi} = n - \frac{1}{2} \tag{A1}$$

and

$$k = \frac{\int_{-\infty}^{\infty}\mathrm{sech}(x)\,\mathrm{d}x}{\int_{-\infty}^{\infty}\sin\left[(2n-1)\int_{-\infty}^{x}\mathrm{sech}(x')\,\mathrm{d}x'\right]\mathrm{d}x}$$

$$= \frac{\pi}{\int_{-\infty}^{\infty}\sin[(2n-1)2\tan^{-1}(e^x)]\,\mathrm{d}x} = \frac{\pi}{\pi} = 1\,. \tag{A2}$$

In spite of the fact that the straight line $\eta = k(A - A_c)$ is obtained with an approximation procedure assuming small eigenvalues, for this particular choice of potential the exact result is valid for all amplitudes, i.e. $\eta = A - (n - \frac{1}{2})$ is regained.

In the asymptotic analysis based on the Bohr–Sommerfeld equation, the single-humped potential is expanded around its maximum. In the case of $Q(x) = \mathrm{sech}(x)$, we obtain $Q \approx (1 - x^2/2)$.

The straight-line solution given below equation (6) ($\eta \cong A - (n - \frac{1}{2})\sqrt{2b}$) then immediately yields the exact result, $\eta = A - (n - \frac{1}{2})$ valid for all amplitudes, in spite of the fact that the analysis in this case is based on the assumption of large amplitudes.

Thus, for this particular potential, the straight lines describing the asymptotic variation of the eigenvalue for small and large amplitudes A, respectively, coincide and are identical to the exact result.

References

[1] Drazin P G and Johnson R S 1989 *Solitons: an Introduction* (Cambridge: Cambridge University Press)
[2] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[3] Boffeta G and Osborne A R 1992 *J. Comput. Phys.* **102** 252
[4] Osborne A R 1993 *Phys. Lett.* **176A** 75
[5] Lewis Z W 1985 *Phys. Lett.* **112A** 99
[6] Kaup D J 1977 *Phys. Rev.* A **16** 704
[7] Desaix M, Anderson D and Lisak M 1994 *Phys. Rev.* E **50** 2253
[8] Kaup D J and Malomed B A 1995 *Physica* **84D** 319
[9] Desaix M, Anderson D, Lisak M and Quiroga-Teixeiro M L *Phys. Lett.* to appear
[10] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons, the Inverse Scattering Method* (New York: Consultants Bureau)
[11] Zakharov V E and Shabat A B 1972 *Sov. Phys.–JETP* **34** 62
[12] Kivshar Y S 1989 *J. Phys. A: Math. Gen.* **22** 337
[13] Matsumoto M, Hasegawa A and Kodama Y 1994 *Opt. Lett.* **19** 1019